TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 198, 1974

RECURRENT RANDOM WALK OF AN INFINITE PARTICLE SYSTEM

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ABSTRACT. Let $p(x,y)$ be the transition function for a symmetric irreducible recurrent Markov chain on a countable set S. Let η_t be the infinite particle system on S moving according to simple exclusion interaction with the one particle motion determined by p. Assume that p is such that any two particles moving independently on S will sooner or later meet. Then it is shown that every invariant measure for η_t is a convex combination of Bernoulli product measures μ_α on $\{0,1\}^S$ with density $0 \le \alpha = \mu[\eta(x) = 1] \le 1$. Ergodic theorems are proved concerning the convergence of the system to one of the μ_α.

1. Introduction. Let S be an arbitrary countable set. On S we suppose given the transition function $p(x,y)$ of an irreducible Markov chain. Further p is assumed symmetric, i.e. $p(x,y) = p(y,x)$. Let $X = \{0,1\}^S$, with the product topology. Here is an intuitive description of the random walk of a system of particles on S. Let $\eta \in X$ describe an initial configuration of particles, with the interpretation that $\eta(x) = 1$, or 0, according as the site x in S is occupied by a single particle, or vacant. Each particle now waits a random (exponentially distributed) time with mean one. At the end of this holding time the particle attempts to jump from its site (let us call it x) to the site y with probability $p(x,y)$. The jump takes place if and only if the site y is vacant at that instant. It has been shown by T. M. Liggett [4] that there exists a strong Markov process η_t, $t \ge 0$, with state space X, which corresponds to this intuitive description. Let $S(t)$ denote the semigroup of this process. It acts on probability measures μ on X in the usual way, i.e. $\nu = \mu S(t)$ means that

$$\int_X \nu(d\eta) f(\eta) = \int_X \mu(d\eta)[S(t)f](\eta)$$

for all continuous functions f on X. A probability measure μ on X (with the usual σ-algebra of subsets) is called an *invariant* or *equilibrium measure* if $\mu S(t) = \mu$ for all $t \ge 0$. The problem is, first of all, to characterize all the invariant measures; secondly, to obtain ergodic theorems. This means describing, for each invariant measure μ, the class of probability measures ν on X such that $\nu S(t) \Rightarrow \mu$, as $t \to +\infty$. By $\Rightarrow$ we mean weak convergence, equivalently, since X is compact, convergence of the finite dimensional distributions.

Received by the editors February 26, 1973.

AMS (MOS) subject classifications (1970). Primary 60K35; Secondary 47A35.

Key words and phrases. Infinite particle systems, invariant measures, random walk, ergodic theorems.

[1] Research supported by the Mittag-Leffler Institute, Djursholm, Sweden.

When S is a *finite* set these problems are easy and were solved in [8]. (In this case it suffices that $p(x,y)$ be doubly stochastic rather than symmetric.) Therefore we assume that S is infinite in the sequel. When $p(x,y)$ is *transient* the above problems were completely solved by Liggett [5]. Let $\mathcal{I}$ denote the compact convex set of invariant probability measures. Liggett showed that the extreme points of $\mathcal{I}$ are in a one-to-one correspondence with the class $\mathcal{H}$ of functions f on S such that $0 \leq f \leq 1$, and

$$\sum_{y \in S} p(x,y) f(y) = f(x), \qquad x \in S.$$

In particular, when $p(x,y)$ is such that $\mathcal{H}$ consists only of the constants α, $0 \leq \alpha \leq 1$, $\mathcal{I}$ is the well-known class $\mathcal{P}$ of symmetric probability measures on S, whose extreme points are the product measures μ_α, with density α, $0 \leq \alpha \leq 1$, i.e. $\mu_\alpha[\eta\colon \eta(x_i) = 1, i = 1, 2, \ldots n] = \alpha^n$, when $x_1, x_2, \ldots, x_n$ are n distinct sites of S. Liggett conjectured that this state of affairs persists when $p(x,y)$ is *recurrent*. This will be proved in Theorem 2 below, under the following additional condition.

(C) *Let two Markov chains move according to $p(x,y)$ in the following way: at each unit time one of them is selected at random and makes a transition according to $p(\cdot, \cdot)$. Then they will sooner or later occupy the same point of S, with probability one.*

It is nontrivial to show that condition (C) may fail to hold. This is done in a companion paper to this one by T. M. Liggett [6]. There Liggett proves the above conjecture and related ergodic theorems in the case when $p(x,y)$ is recurrent and (C) fails. His methods are different from ours, and do not apply when (C) holds.

In Theorem 3 we give a necessary and sufficient condition for a probability measure μ on X to satisfy $\mu S(t) \Rightarrow \mu_\alpha$. Theorem 4 contains a simpler sufficient condition, communicated to by me Liggett. Theorems 5 and 6 contain a complete account of what happens when S is a countable Abelian group. The random walk defined by the group invariant transition function $p(x,y)$ may be transient or recurrent. The transient case has been settled in [5]. Theorems 5 and 6 simply state that all the results are the same in the recurrent case. Theorem 6 depends crucially on Theorem 4. We conclude by applying the results to simple random walk on the integers.

2. **The finite particle system.** Just as in the study of Liggett [5], the ergodic theory of the infinite particle system η_t, $t \geq 0$, with $\sum \eta_0(x) = +\infty$, can be reduced to the special case when there are only N particles. Then

$$N = \sum_{x \in S} \eta_0(x) = \sum_{x \in S} \eta_t(x), \qquad t \geq 0.$$

It is possible to give a contracted description of this process, by looking only at the positions of the N particles. Let S^N be the product of N copies of S, D the set of all $\vec{x} = (x_1, x_2, \ldots, x_N) \in S^N$ such that two or more of the coordinates x_i are equal, and let $T_N = S^N \backslash D$ Following Liggett [5, §3], we define the transition operator V_N on T_N by

$$V_N f(\vec{x}) = \frac{1}{N}\left[\sum_{i=1}^{N}\sum_{\substack{j=1\\ i\neq j}}^{N} p(x_i,x_j)\right] f(\vec{x})$$

$$(2.1) \qquad + \frac{1}{N}\sum_{i=1}^{N}\sum_{u\neq x_j \text{ for } j\neq i} p(x_i,u) f(x_1,x_2,\cdots,x_{i-1},u,x_{i+1},\cdots,x_N), \qquad \vec{x}\in T_N.$$

Thus V_N corresponds to the transition: select one of the N particles at random and let it jump according to $p(x,y)$, but only if there is no particle at y. Finally, let V_N^t, $t \geq 0$, be the semigroup of transition operators on T_N defined by

$$(2.2) \qquad V_N^t = e^{-Nt}\sum_{k=0}^{\infty}\frac{(Nt)^k}{k!}V_N^k = \exp tN[V_N - I].$$

This is then the contracted description of η_t, $t \geq 0$, when $\sum \eta_0(x) = N$; as there are N particles, the expected number of attempted jumps in time t is Nt.

Let $\mathcal{B}_N$ be the space of real functions f on T_N with norm

$$\|f\| = \sup_{\vec{x}\in T_N}|f(\vec{x})| \leq 1,$$

and let $\mathcal{S}_N \subset \mathcal{B}_N$ be the set of symmetric functions in $\mathcal{B}_N$. We would like to show that $V_N^t f(\vec{x}) - V_N^t f(\vec{y})$ is small for large t for $\vec{x} \neq \vec{y}$ in T_N. This is not true if $f \in \mathcal{B}_N \backslash \mathcal{S}_N$: Let $S = \mathbf{Z}$, $N = 2$, $p(x,y) = \frac{1}{2}$ if $|x - y| = 1$, and define $f(\vec{x})$ as $+1$ when $x_1 < x_2$, and 0 when $x_1 > x_2$. Then $V_2^t f(\vec{x}) = 1$ for all $t \geq 0$ and all $\vec{x}$ such that $x_1 < x_2$, and equal to zero for all $t \geq 0$ and all $\vec{x}$ such that $x_1 > x_2$. Fortunately we shall need only the following result.

Theorem 1. *For all $N \geq 1$, and all $\vec{x}, \vec{y} \in T_N$*

$$\lim_{t\to\infty}\sup_{f\in\mathcal{S}_N}|V_N^t f(\vec{x}) - V_N^t f(\vec{y})| = 0.$$

First we observe that it suffices to prove the theorem in the case when $\vec{x}$ differs from $\vec{y}$ only in one coordinate. Hence we may assume $x_1 \neq y_1$, $x_k = y_k$ for $k = 2, 3, \ldots, N$. The proof will depend on a delicate coupling of two Markov processes, each with transition function V_N^t. This method was inspired by R. Holley's use of coupling to prove ergodic theorems for somewhat similar stochastic time evolution models [2], [3].

Let X_t and Y_t be two independent Markov processes on T_N, each with transition semigroup V_N^t, $N \geq 1$. Assume the initial conditions $X_0 = \vec{x}$, $Y_0 = \vec{y}$, where $x_k = y_k$ except for $k = 1$. We shall construct a "coupling" of these two processes, as follows:

(a) The coupled process is a Markov process $Z_t = (X'_t, Y'_t)$ with state space $T_N \times T_N$ and with initial condition $Z_0 = (\vec{x}, \vec{y})$.

(b) For each $t \geq 0$, X'_t has the same probability distribution as X_t, and Y'_t as Y_t.

(c) Let A_t denote the set of coordinates of X'_t (i.e. if the vector $X'_t = (x_1, x_2, \ldots, x_N)$ then A_t is the unordered set $\{x_1, x_2, \ldots, x_N\}$ consisting of N distinct elements of S). Similarly B_t is the set of coordinates of Y'_t. Then there exists a positive random variable τ, finite with probability one, such that $A_t = B_t$ for all $t \geq \tau$.

Using properties (a), (b), (c) the proof of Theorem 1 is easy. Let $E^{(\vec{x},\vec{y})}$ denote expectation with respect to the above process Z_t on $T_N \times T_N$. Then (a) and (b) imply

$$V_N^t f(\vec{x}) - V_N^t f(\vec{y}) = E^{(\vec{x},\vec{y})}[f(X'_t) - f(Y'_t)]$$

for all $t \geq 0$, and $f \in \mathcal{B}_N$. If in addition $f \in \mathcal{S}_N$, then $f(\vec{x})$ depends only on the set $A = \{x_1, x_2, \ldots, x_N\}$. So we may write $f(\vec{x}) = \tilde{f}(A)$. Using (c) one has for $f \in \mathcal{S}_N$

$$\begin{aligned} |V_N^t f(\vec{x}) - V_N^t f(\vec{y})| &= |E^{(\vec{x},\vec{y})}[\tilde{f}(A_t) - \tilde{f}(B_t)]| \\ &= |E^{(\vec{x},\vec{y})}[\tilde{f}(A_t) - \tilde{f}(B_t); \tau > t]| \leq 2P^{(\vec{x},\vec{y})}[\tau > t]. \end{aligned}$$

Theorem 1 follows, since $\tau < \infty$ with probability one. When $N = 1$ it may be noted that the conclusion of Theorem 1 is correct also without condition (C). This follows from the ergodic theorem of Orey [7].

Returning to the construction of the coupled process $Z_t = (X'_t, Y'_t)$, think of X'_t and Y'_t as the time evolutions of two systems of N particles, in two containers S_1, S_2, each a copy of S. First we couple the two containers S_1, S_2. Each site $x \in S_1$ is coupled to the site $x \in S_2$ so that a random exponential (mean 1) clock rings simultaneously at $x \in S_1$ and at $x \in S_2$. This is done, independently, for each $x \in S$. The law of the random time evolutions in S_1 and S_2 is this: when the random clock rings at x there are three possibilities:

(i) x is occupied in S_1 and in S_2. Then both particles try to move according to $p(x,y)$, and this motion is also coupled in the sense that they both try to move to the same point y, with probability $p(x,y)$. In each container the jump to y takes place if and only if y is vacant.

(ii) x is occupied only in one container, say in S_1. Then the particle at $x \in S_1$ tries to jump to y, with probability $p(x,y)$. The jump takes place only if y is vacant.

(iii) x is vacant in S_1 and S_2. Nothing happens.

It should be clear from (i), (ii), (iii) that (a) and (b) hold, i.e. Z_t is Markovian, and the marginal processes X'_t and Y'_t are also Markov processes with the same joint distributions as X_t and Y_t. To verify (c), suppose that at a certain time t, the coordinate sets A_t and B_t satisfy $A_t = C_t \cup \xi_t$, $B_t = C_t \cup \eta_t$, where C_t is a set of cardinality $N - 1$ which is the same set in S_1 and S_2. This is the case at $t = 0$. To show this is true for all later $s \geqslant t$ let $C_t = C$, $\xi_t = x$, $\eta_t = y$, $x \neq y$. Thus

$x \in S_1$ and $y \in S_2$ are the positions of the "odd" particles. We shall now compute the infinitesimal generator of (ξ_s, η_s), $s \geq t$, by considering all possible changes in the interval $[t, t + \Delta]$.

(α) The clock rings at x in time Δ. This results in a jump from x to a in container S_1, with probability $\Delta p(x,a)$, for any $a \in S \backslash C$. Note that nothing happens in S_2 since x is vacant there. Thus we have the following contribution to the generator

$$(x,y) \to (a,y) \text{ with probability } \Delta p(x,a), \qquad a \in S\backslash C.$$

(β) The clock rings at y in time Δ. Just as above we get

$$(x,y) \to (x,b) \text{ with probability } \Delta p(y,b), \qquad b \in S\backslash C.$$

(γ) The clock rings at a point $c \in C$, in time Δ. Note that c is occupied in both containers. Yet nothing happens to the pair of odd particles unless there is a jump from c to x in S_2 or from c to y in S_1. In the first case the coordinates of the "odd" pair (ξ_t, η_t) change from (x, y) to (c, y); in the second case from (x, y) to (x, c). Schematically we have

$$(x,y) \to (c,y) \text{ with probability } \Delta p(c,x), \qquad c \in C,$$

$$(x,y) \to (x,c) \text{ with probability } \Delta p(c,y), \qquad c \in C.$$

First of all this classification shows that the coordinate sets of the two processes X'_t and Y'_t will never differ by more than one pair of points if this is true at time 0. To verify (c) combine the specific expressions for the generator in cases (α), (β), (γ) making use of the symmetry of $p(x,y)$ to write $p(c,x) = p(x,c)$, $p(c,y) = p(y,c)$ in part (γ). The result is that in time Δ a change of $(\xi_t, \eta_t) = (x,y)$ to (x, a) has probability $\approx \Delta p(y,a)$ for all $a \in S$ and to (a, y) the probability is $\approx \Delta p(x,a)$ for all $a \in S$. But this is just the statement that the infinitesimal generator of (ξ_t, η_t) is exactly the same as of a pair of independent Markov processes with transition semigroup V_1^t (whose generator is the sum of the generators $p(x,y) - \delta(x,y)$ for each process). Since this generator is a bounded operator it clearly defines the process (ξ_t, η_t) uniquely up to the first time τ, when $\xi_t = \eta_t$. This stopping time is finite with probability one since the pair (ξ_t, η_t) observed only at instants of change behaves just like the discrete time Markov chain in condition (C). After time τ we have identical coordinate sets in S_1 and S_2, i.e. $A_t = B_t$ for all $t \geq \tau$. This is clear from (i), (ii), (iii) above.

3. **The infinite particle system.** For an arbitrary probability measure μ on X, define the system of its correlation functions $\rho^{(N)}$, $N \geq 1$, by

$$\rho^{(N)}(\vec{x}) = \mu[\eta \mid \eta(x_1) = \eta(x_2) = \cdots = \eta(x_N) = 1],$$

$$\vec{x} = (x_1, \ldots, x_N) \in T_N.$$

Note that each $\rho^{(N)}$ is a symmetric function on T_N. If $\rho_t^{(N)}$ are the correlation functions of a family of measures μ_t, $t \geq 0$, and if $\mu_t = \mu_0 S(t)$, $t \geq 0$, then the correlations satisfy

$$\rho_t^{(N)}(\vec{x}) = V_N^t \rho_0^{(N)}(\vec{x}), \qquad N \geq 1, \vec{x} \in T_N. \tag{3.1}$$

This basic result occurs in [5] and [8]. It is easily derived by checking that the correlations satisfy the diffusion equation

$$\frac{\partial}{\partial t} \rho_t^{(N)}(\vec{x}) = N(V_N - I)\rho_t^{(N)}(\vec{x}), \qquad t \geq 0, \vec{x} \in T_N. \tag{3.2}$$

Since the operators $V_N - I$ are bounded, (3.1) gives the unique solution. It is well known that a probability measure μ on X is uniquely determined by its correlations. This fact together with (3.1) and Theorem 1 yields the ergodic theory for the infinite particle system η_t, $t \geq 0$.

Theorem 2. *The set of all equilibrium measures is $\mathcal{P}$.*

Proof. The set $\mathcal{P}$ of symmetric measures is the closed convex hull of the product measures μ_α on X. These have constant correlation functions. Hence (3.1) implies that $\mathcal{P} \subset \mathcal{I}$. Conversely suppose $\mu \in \mathcal{I}$ with correlations $\rho^{(N)}$. Then if $\rho_t^{(N)}$ are the correlations of $\mu S(t)$, we have $\rho_t^{(N)} = \rho^{(N)}$. Theorem 1 applied to (3.1) gives

$$\begin{aligned} |\rho^{(N)}(\vec{x}) - \rho^{(N)}(\vec{y})| &= |V_N^t \rho^{(N)}(\vec{x}) - V_N^t \rho^{(N)}(\vec{y})| \\ &\leq \sup_{f \in \mathcal{S}_N} |V_N^t f(\vec{x}) - V_N^t f(\vec{y})| \to 0, \quad \text{as } t \to \infty. \end{aligned}$$

Hence each $\rho^{(N)}$ is constant on T_N. Therefore the cylinder set probabilities of μ (and hence μ itself) are invariant under all finite permutations of S. This property characterizes $\mathcal{P}$ by de Finetti's theorem [1].

Theorem 3. *Let μ_α be the product measure on X with density α, $0 \leq \alpha \leq 1$. Then $\mu S(t) \Rightarrow \mu_\alpha$ as $t \to \infty$ if and only if the correlations of μ satisfy*

(a) $\lim_{t\to\infty} V_1^t \rho^{(1)}(x) = \alpha$, $x \in S = T_1$;

(b) $\lim_{t\to\infty} V_2^t \rho^{(2)}(\vec{x}) = \alpha^2$, $\vec{x} \in T_2$.

Proof. If $\mu S(t) \Rightarrow \mu_\alpha$ then $\rho_t^N(\vec{x}) \to \alpha^N$ as $t \to \infty$ for each $N \geq 1$. Thus (3.1) implies the necessity of (a), (b). Suppose now μ is such that (a) and (b) hold. Since S is countable we may find a sequence $t_n \nearrow +\infty$ such that

$$\lim_{n\to\infty} V_N^{t_n} \rho^{(N)}(\vec{x}) = c_N(\vec{x}) \tag{3.3}$$

exists for each $N \geq 1$, and all $\vec{x} \in T_N$. It follows from Theorem 1 that each c_N is constant on T_N. Now the c_N are limits of correlation functions of a sequence of probability measures. By compactness of the set of probability measures on X, we know that $\{c_N\}$ is the sequence of correlations of a probability measure ν on

X. As mentioned in the proof of Theorem 2, that measure is in $\mathcal{P}$. Hence it is of the form

$$\nu = \int_0^1 \mu_\gamma dF(\gamma).$$

Thus

$$c_N = \int_0^1 \gamma^N dF(\gamma), \qquad N \geq 1,$$

for some probability distribution F on $[0, 1]$. But by hypotheses (a) and (b)

$$c_1 = \int_0^1 \gamma\, dF(\gamma) = \alpha, \qquad c_2 = \int_0^1 \gamma^2 dF(\gamma) = \alpha^2.$$

This implies that F concentrates its mass at $\gamma = \alpha$. Hence $c_N = \alpha^N$ for all $N \geq 1$. But we have proved this independently of how the subsequence t_n in (3.3) was chosen. Therefore

$$\lim_{t\to\infty} V_N^t \rho^{(N)}(\vec{x}) = \alpha^N, \qquad N \geq 1, \vec{x} \in T_N.$$

This implies that $\mu S(t) \Rightarrow \mu_\alpha$.

Condition (b) in Theorem 3 is difficult to verify in practice, because of the interaction of the two-particle system moving according to V_2^t. It would be simpler to deal with the independent two-particle system represented by the operator $U_2^t = \exp 2t(U_2 - I)$, where

$$U_2 f(\vec{x}) = \tfrac{1}{2} \sum_u [p(x_1, u) f(u, x_2) + p(x_2, u) f(x_1, u)], \qquad \vec{x} \in S^2.$$

The following result is due to T. Liggett.

Theorem 4. *The conditions in Theorem* 3 *remain sufficient for* $\mu S(t) \Rightarrow \mu_\alpha$, *if* V_2^t *in condition* (b) *is replaced by* U_2^t.

Proof. Suppose that the modified condition (b) holds. Observe that

$$\rho^{(2)}(\vec{x}) = \mu[\eta\colon \eta(x_1) = \eta(x_2) = 1], \qquad \vec{x} = (x_1, x_2) \in S^2$$

is a bounded, symmetric, positive definite function on S^2. Liggett has shown in Lemma 2.7 of [6] that every such function f satisfies

$$V_2^t f(\vec{x}) \leq U_2^t f(\vec{x}), \qquad \vec{x} \in T_2.$$

Therefore

$$\limsup_{t\to\infty} V_2^t \rho^{(2)}(\vec{x}) \leq \alpha^2, \qquad \vec{x} \in T_2.$$

Suppose now that (a) holds, and that for some subsequence $t' \to \infty$, we obtain a limit less than α^2. Just as in the proof of Theorem 3 this would imply

$$\int_0^1 \gamma^2 dF(\gamma) < \left[\int_0^1 \gamma\, dF(\gamma)\right]^2 = \alpha^2,$$

which is impossible. Hence (b) in Theorem 3 holds, which proves Theorem 4.

Remark. Theorem 4 may be expressed, as in Theorem 1.5 of [5], by saying that $\mu S(t) \Rightarrow \mu_\alpha$, provided

$$\lim_{t\to\infty} e^{-t} \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{y\in S} p^{(n)}(x,y)\eta_0(y) = \alpha \tag{3.4}$$

in probability, with respect to μ, for each $x \in S$. Since $p(x,y)$ is recurrent it even suffices, by Orey's ergodic theorem [7], to assume that (3.4) holds for a single point $x \in S$.

Further simplification results if we assume that the probability measure μ is concentrated at a point of X, i.e. that the initial configuration $\eta_0(x)$, $x \in S$, is a given nonrandom assignment of zeros and ones. In this case (3.4) is evidently equivalent to condition (a) in Theorem 3. Therefore (3.4) *is necessary and sufficient for* $\mu S(t) \Rightarrow \mu_\alpha$, whenever μ is concentrated on the single point $\eta_0 \in X$.

4. **Random walk on a group.** Suppose now that S is an additive Abelian group, and that $p(x,y) = p(0,y-x)$ for all $x, y \in S$. In this case condition (C) holds automatically when $p(x,y)$ is recurrent. To show this, let X_t, Y_t be two independent random walks on S, both starting at 0, i.e. processes with transition semigroup V_1^t, $t \geq 0$. Then $X_t - Y_t$ is exactly the same process as X_{2t}. Hence $X_t - Y_t = 0$ infinitely often with probability one. Therefore (X_t, Y_t) visits the diagonal $D \subset S^2$ with probability one from (0, 0). As $p(x,y)$ is irreducible the same is true for any starting point. Therefore condition (C) holds. Thus Theorem 2 holds. Since Liggett ([5]) has proved the corresponding fact in the transient case we have

Theorem 5. *In the group invariant case, when p is symmetric and irreducible, the set of equilibrium measures is always* $\mathcal{P}$.

In the transient case Liggett has proved [5] that $\mu S(t) \Rightarrow \mu_\alpha$ for every ergodic measure μ with density α, which is invariant under translation by arbitrary elements of the group S. This is also true in the recurrent case.

Theorem 6. *Suppose that* μ *is a translation invariant ergodic probability measure on* X, *with density* $\mu[\eta: \eta(x) = 1] = \alpha$. *Let* $p(\cdot,\cdot)$ *be symmetric, irreducible, group invariant, and either recurrent or transient. Then* $\mu S(t) \Rightarrow \mu_\alpha$.

Proof. The proof is exactly that given by Liggett in the transient case; see Theorem 5.6 in [5]. It consists in verifying the conditions in Theorem 4 above and is independent of whether the random walk is recurrent or transient.

Example. Let $S = \mathbf{Z}$, the integers, and $p(x,y) = \frac{1}{2}$ if $|x-y| = 1$, and 0 otherwise. Then the equilibrium measure μ_α is approached by the time evolution if the initial sequence $\eta_0(x)$, $x \in \mathbf{Z}$, is any stationary 0, 1 valued ergodic process with mean α. Suppose now that $\eta_0(x)$, $x \in \mathbf{Z}$, is a given nonrandom sequence of zeros and ones. By the remark following the proof of Theorem 4 we get convergence to μ_α if and only if

$$e^{-t}\sum_{n=0}^{\infty}\frac{t^n}{n!}\sum_{k\in\mathbf{Z}}p^{(n)}(0,k)\eta_0(k)=\sum_{k=-\infty}^{\infty}e^{-t}I_{|k|}(t)\eta_0(k)$$

tends to α as t tends to infinity. Here I_k denotes a familiar Bessel function. By direct computation, or by use of a local central limit theorem, this can be proved for every initial configuration with density α, i.e. satisfying

$$\lim_{n\to\infty}\frac{1}{2n+1}\sum_{k=-n}^{n}\eta_0(k)=\alpha.$$

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